Lecture 7

Wednesday, October 16, 2019 5:24 AM

<u>Recall</u>. • $f:(X,d)\to(\Omega,\rho)$ is <u>cont.</u> if $\forall a, \varepsilon>0\ \exists\,\delta>0$ s.t. $\rho(f(x),f(a))<\varepsilon$ when $d(x,a)<\delta$.

• Equivalently, $\forall\,\Delta\subseteq\Omega$ open $\Rightarrow f^{-1}(\Delta)\subseteq X$ is open.

<u>Basic Props</u>. ① If $f,g:X\to\Omega$ are cont. $\Rightarrow$ $f+g$, fg are cont.
$f/g: X\setminus\{g=0\}\to\Omega$ is also cont.

② If $f:X\to Y$, $g:Y\to\Omega$ are cont. $\Rightarrow g\circ f: X\to\Omega$ is cont.

Pfs left as <u>Ex.</u> (for ②, use 2^{nd} char. of continuity.)

<u>Def</u>. ① $f:X\to\Omega$ is <u>uniformly continuous</u> if $\forall\varepsilon>0\ \exists\,\delta>0$ s.t. $\rho(f(x),f(y))<\varepsilon$ when $d(x,y)<\delta$.

② f is <u>Lipschitz cont.</u> if $\exists\, C>0$ s.t. $\rho(f(x),f(y))\le C\,d(x,y)$.

Clearly: f Lipschitz $\Rightarrow$ f unif cont. $\Rightarrow$ f cont.

• <u>Important Ex</u>. Let $A\subseteq X$, and define $d(\cdot,A): X\to\mathbb{R}_+$ by
$d(x,A):=\inf\limits_{y\in A} d(x,y)$. Then, $d(\cdot,A)$ is Lipschitz w/ $C=1$.

<u>Pf</u>. Pick $\varepsilon>0$, $\exists\, a\in A$ s.t. $d(x,a)<d(x,A)+\varepsilon$ $\Rightarrow$

$$d(y,A)-d(x,A) < d(y,a)-(d(x,a)-\varepsilon) \overset{\Delta\text{-ineq.}}{\le} d(y,x)+d(x,a)-d(x,a)+\varepsilon = d(y,x)+\varepsilon.$$

$\Rightarrow d(y,A)-d(x,A)\le d(x,y)$ since ε arbitrary.

Same argument $\Rightarrow$ $d(x,A)-d(y,A)\le d(x,y)$ $\Rightarrow$ $|d(x,A)-d(y,A)|\le d(x,y)$. ∎

<u>Thm</u>. Let $f:X\to\Omega$ be cont.

(i) If $K\subseteq X$ is compact $\Rightarrow$ $f(K)\subseteq\Omega$ is compact.

(ii) If $A\subseteq X$ is connected $\Rightarrow$ $f(A)\subseteq\Omega$ is connected.

<u>Pf</u>: (i) Let $\{\Delta_\alpha\}_{\alpha\in I}$ be open cover of $f(K)$. Then $\{f^{-1}(\Delta_\alpha)\}_{\alpha\in I}$ is open cover of K. By assumption, $\exists$ finite subcover $K\subseteq\bigcup\limits_{k=1}^{n} f^{-1}(\Delta_{\alpha_k})$

But then $f(K)\subseteq\bigcup\limits_{k=1}^{n}\Delta_{\alpha_k}$; finite subcover $\Rightarrow$ $f(K)$ compact.

But then $f(K) \subseteq \bigcup_{k=1}^{N} \Delta_{x_k}$; finite subcover $\Rightarrow f(K)$ compact.

(ii) Suppose $f(A)$ not connected. $\Rightarrow \exists\, B \subseteq \Omega$ open + closed s.t. $B \cap f(A) \neq \emptyset, f(A)$. But then $f^{-1}(B) \subseteq X$ is open + closed by cont. $\underbrace{f^{-1}(B) \cap A \neq \emptyset}_{\exists a \in A \text{ s.t. } f(a) \in B}$ and $\underbrace{f^{-1}(B) \cap A \neq A}_{f(A) \not\subseteq B}$.

This is $\cancel{\exists}$ since A is connected. $\blacksquare$

- Important consequence:

<u>Thm 2</u> If $f: X \to \mathbb{R}$ is cont., $K \subseteq X$ compact, then $\exists\, x_1, x_2 \in K$ s.t.

$$\sup_{x \in K} f(x) = f(x_1) \; ; \; \inf_{x \in K} f(x) = f(x_2).$$

<u>Pf.</u> By Thm 1, $f(K) \subseteq \mathbb{R}$ is compact. By Heine-Borel, $f(K)$ is closed and bdd. $\Rightarrow$ $y_1 = \inf_{y \in f(K)} y$, $y_2 = \sup_{y \in f(K)} y$

belong to $f(K) \Rightarrow \exists\, x_1, x_2 \in K$ s.t. $f(x_1) = y_1$, $f(x_2) = y_2$. $\blacksquare$

- Very important results:

<u>Thm 3</u>. If $f: X \to \Omega$ cont. and X compact $\Rightarrow f$ is unif. cont.

<u>Pf.</u> Pick $\varepsilon > 0$. For every $a \in X$ $\exists\, \delta_a \geq 0$ s.t. $\rho(f(x), f(a)) < \varepsilon/2$ when $d(x,a) < \delta_a$. Consider $G_a := B(a, \delta_a)$ open. If $x, y \in G_a \Rightarrow \rho(f(x), f(y)) \overset{\Delta\text{-ineq.}}{\leq} \rho(f(x), f(a)) + \rho(f(a), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Now, $\{G_a\}_{a \in X}$ is an open cover of X. Since X compact $\Leftrightarrow$ seq. compact, by Lebesgue's Covering Lemma $\boxed{\exists\, \delta > 0}$ $\forall b \in X$ $B(b, \delta) \subseteq G_a = B(a, \delta_a)$ for some a. But then if $d(x,b) < \delta \Rightarrow x, b \in G_a \Rightarrow \rho(f(x), f(b)) < \varepsilon$. $\Rightarrow$

if $d(x,b) < \delta \Rightarrow x, b \in G_a \Rightarrow \rho(f(x), f(b)) < \varepsilon. \Rightarrow$
f unif. cont. ▨

<u>Thm 4</u>. If $F \subseteq X$ closed, $K \subseteq X$ compact, $F \cap K = \emptyset$
$\Rightarrow d(F,K) := \inf_{x \in F,\, y \in K} d(x,y) > 0$.

<u>Pf</u>. Observe $d(F,K) = \inf_{x \in K} d(x,F)$. Now, $f: X \to \Omega$ given by $f(x) = d(x,F)$ is cont. (Lipschitz). Since K compact, Thm 2 $\Rightarrow \exists x_0 \in K$ s.t. $f(x_0) = d(x_0, F) = \inf_{x \in K} d(x,F) = d(F,K)$.

But it is easy to see that $d(x_0, F) = 0 \Leftrightarrow x_0$ is a limit point of F. Since F is closed, if x_0 were limit pt of F then $x_0 \in F$. But $F \cap K = \emptyset \Rightarrow d(x_0, F) > 0$.
▨